Permanents of Circulants: a Transfer Matrix Approach (Expanded Version)*

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Abstract

Calculating the permanent of a $(0,1)$ matrix is a $\#P$-complete problem but there are some classes of *structured* matrices for which the permanent is calculable in polynomial time. The most well-known example is the *fixed-jump* $(0,1)$ circulant matrix which, using algebraic techniques, was shown by Minc to satisfy a constant-coefficient fixed-order recurrence relation.

In this note we show how, by interpreting the problem as calculating the number of cycle-covers in a directed circulant *graph,* it is straightforward to reprove Minc's result using combinatorial methods. This is a two step process: the first step is to show that the cycle-covers of directed circulant graphs can be evaluated using a *transfer matrix* argument. The second is to show that the associated transfer matrices, while very large, actually have much smaller characteristic polynomials than would a-priori be expected.

An important consequence of this new viewpoint is that, in combination with a new recursive decomposition of circulant-graphs, it permits extending Minc's result to calculating the permanent of the much larger class of circulant matrices with *non-fixed* (but linear) jumps. It also permits us to count other types of structures in circulant graphs, e.g., Hamiltonian Cycles.

1 Introduction

Definition 1 *Let $A = (a_{i,j})$ be an $n \times n$ matrix. Let S_n be the set of permutations of the integers $[1, \ldots, n]$. The* permanent *of A is*

$$\mathrm{Perm}(A) = \sum_{\pi \in S_n} \prod_{i=1}^{n} a_{i,\pi(i)} \quad \textit{where} \quad \pi = [\pi(1), \ldots, \pi(n)]. \tag{1}$$

If A is a $(0,1)$ matrix, then A can be interpreted as the adjacency matrix of some directed graph G and $\mathrm{Perm}(A)$ is the number of *directed cycle-covers* in G, where a directed cycle-cover is a collection of disjoint cycles that cover all of the vertices in the graph. Alternatively, A can be interpreted as the adjacency

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matrix of a bipartite graph $\bar{G}$, in which case $\mathrm{Perm}(A)$ is the number of *perfect-matchings* in $\bar{G}$. The permanent is a classic well-studied combinatorial object (see the book and later survey by Minc[14, 17]).

Calculating the permanent of a $(0, 1)$ matrix is a $\#P$-Complete problem [20] even when A is restricted to have only 3 non-zero entries per row [8]. The best known algorithm for calculating a general permanent is a straightforward inclusion-exclusion technique due to Ryser [14] running in $\Theta(n2^n)$ time and polynomial space. By allowing super-polynomial space, Bax and Franklin [1] developed a slightly faster (although still exponential) algorithm for the $(0, 1)$ case. For non-exact calculation Jerrum, Sinclair and Vigoda [12] have developed a fully polynomial approximation scheme for *approximating* the permanent of nonnegative matrices.

On the other hand, for certain special structured classes of matrices one can *exactly* calculate the permanent in "polynomial time". The most studied example of such a class is probably the *circulant matrices,* which, as discussed in [7], can be thought of as the borderline between the easy and hard cases.

An $n \times n$ circulant matrix $A = (a_{i,j})$ (see Figures 1 (a) and (c)) is defined by specifying its first row; the $(i+1)^{\mathrm{st}}$ row is a cyclic shift i units to the right of the first row, i.e., $a_{i,j} = a_{1,1+(n+j-i) \bmod n}$. Let P_n denote the $(0, 1)$ $n \times n$ matrix with **1**s in positions $(i, i + 1)$, $i = 1, \ldots, n - 1$, and $(n, 1)$ and **0**s everywhere else. Many of the early papers on this topic express circulant matrices in the form

$$A_n = a_1 P_n^{s_1} + a_2 P_n^{s_2} + \cdots + a_k P_n^{s_k} \tag{2}$$

where $0 \leq s_1 < s_2 < \cdots < s_k < n$ and $a_i = a_{1,s_i+1}$.

The first major result on permanents of $(0, 1)$ circulants was due to Metropolis, Stein and Stein [13]. Let $k > 0$ be fixed and $A_{n,k} = \sum_{i=0}^{k-1} P_n^i$, be the $n \times n$ circulant matrix whose first row is composed of **1**s in its first k columns and **0**s everywhere else. Then [13] showed that, as a function of n, $\mathrm{Perm}(A_{n,k})$ satisfies a fixed order constant-coefficient recurrence relation in n and therefore, could be calculated in polynomial time in n (after a superpolynomial "start-up cost" in k for deriving the recurrence relation).

This result was greatly improved by Minc who showed that it was only a very special case of a general rule. Let $0 \leq s_1 < s_2 < \cdots < s_k < n$ be *any* fixed sequence and set $A_n = A_n(s_1, \ldots, s_k) = P_n^{s_1} + P_n^{s_2} + \cdots + P_n^{s_k}$. In [15, 16] Minc proved that $\mathrm{Perm}(A_n)$ always satisfies a constant-coefficient recurrence relation in n of order $2^{s_k} - 1$. Minc's theorem was proven by manipulating algebraic properties of A_n. Note, that as mentioned by Minc, this result is difficult to apply for large s_k since, in order to derive the coefficients of the recurrence relation it is first necessary to evaluate $\mathrm{Perm}(A_n)$ for $n \leq 2(2^{s_k} - 1)$ and, using Ryser's algorithm, this requires $\Omega\left(2^{2^{s_k}}\right)$ time.

Later Codenotti, Resta and various coauthors improved these results in various ways; e.g. in [2] showing how to evaluate *sparse* circulant matrices of size ≤ 200; in [4, 5] showing that the permanents of circulants with only three **1**s per row can be evaluated in polynomial time; in [6] showing how the permanents of some special sparse circulants can be expressed in terms of determinants and are therefore solvable in polynomial time; in [2] showing that the permanents

of *dense* circulants are hard to calculate and in [7] that even approximating the permanent of an arbitrary circulant modulo a prime p is "hard" unless $\mathbf{P}^{\#\mathbf{P}} = \mathbf{BPP}$.

In this paper we return to the original problem of Minc. Our first main result will be to show that if circulant *matrix* $A_n(s_1, \ldots, s_k)$ is interpreted as the adjacency matrix of a directed circulant *graph* C_n, then counting the number of cycle-covers of C_n using a *transfer matrix* approach immediately reproves Minc's result. In addition to rederiving Minc's original result using a combinatorial rather than algebraic proof this new technique permits us extend the result to a much larger set of circulant graphs. It will also permit us to address other problems, e.g., counting Hamiltonian cycles in circulant graphs, which at first might seem unrelated. To explain, we first need to introduce some notation.

Definition 2 *See Figure 1. Let $C_n^{s_1,s_2,\cdots,s_k}$. be the n-node* directed circulant graph *with jumps $S = \{s_1, s_2, \ldots s_k\}$. (Note that this definition permits negative s_i.) Formally,*

$$C_n^{s_1,s_2,\ldots,s_k} = (V(n), E_C(n))$$

where

$$V(n) = \{0, 1, \ldots, n-1\}$$

and

$$E_C(n) = \Big\{(i,j) : (j-i) \bmod n \in S\Big\}.$$

Note: we will assume that S contains at least one non-negative s_i since, if all the s_i were negative, we could multiply them by -1 and get an isomorphic graph. Also, we will often write C_n as shorthand for $C_n^{s_1,s_2,\cdots,s_k}$.

Let $G = (V, E)$ be a graph, $T \subseteq E$ and $v \in V$. Define $\mathtt{ID}_T(v)$ to be the *indegree* of v in graph (V, T) and $\mathtt{OD}_T(v)$ to be the *outdegree* of v in (V, T). $T \subseteq E$ is a *cycle-cover* of G if

$$\forall v \in V, \quad \mathtt{ID}_T(v) = \mathtt{OD}_T(v) = 1. \tag{3}$$

Definition 3 *Let $S = \{s_1, s_2, \ldots s_k\}$ be given. Set*

$$\mathcal{CC}(n) = \{T \subseteq C_n : \ T \textit{ is a cycle-cover of } C_n\}$$

and

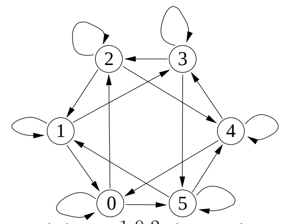
$$T(n) = |\mathcal{CC}(n)| = \textit{No. of cycle-covers of } C_n.$$

Note that, by the standard correspondence mentioned previously, $A_n(s_1, \ldots, s_k)$ is the adjacency matrix of $C_n^{s_1,s_2,\cdots,s_k}$ and $T(n) = \mathrm{Perm}(A_n(s_1, \ldots, s_k))$. So, calculating $T(n)$ is equivalent to calculating permanents of $A_n(s_1, \ldots, s_k)$.

There is also a well-known simple correspondence between cycle covers and permutations. Consider the *directed* complete graph with all n^2 distinct edges on n vertices (self-loops are permitted). Now let S_n be the set of $n!$ permutations on $[0, \ldots, n-1]$. For a fixed permutation $\pi \in S_n$, the set of edges $\bigcup_{i=0}^{n-1}(i, \pi(i))$ is a cycle cover. In the other direction suppose T is a cycle cover. Define π by $\pi(i) = j$ where j is the unique vertex such that $(i, j) \in T$. Then π

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

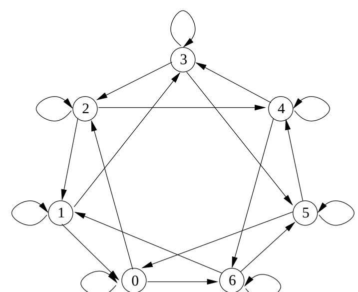
(a) $P_n^{-1} + I + P_n^2$ $(n = 6)$



(b) $C_n^{-1,0,2}$ $(n = 6)$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

(c) $P_n^{-1} + I + P_n^2$ $(n = 7)$



(d) $C_n^{-1,0,2}$ $(n = 7)$

Figure 1: $C_n^{-1,0,2}$: Circulant matrices (a) and (c) are, respectively, the adjacency matrices of circulant graphs $C_n^{-1,0,2}$ in (b) and (d) for $n = 6, 7$.

is a permutation. This is a one-one correspondence between cycle covers and permutations so $T(n)$ counts the number of permutations $\pi \in S_n$ restricted such that $(\pi(i) - i) \bmod n \in S$. For example, if $S = \{1, 2, 3\}$, the number of cycle covers in the corresponding circulant graph $C_n^{1,2,3}$ is equal to the number of permutations π such that $\Big((\pi(i) - i) \bmod n\Big) \in \{1, 2, 3\}$. In fact, in [19, Sec 4.7], Stanley shows that, for fixed S, the number of such permutations satisfies a recurrence relation, giving an alternative derivation of Minc's result for this special case (but without the bound on the order of the recurrence relation given in [15, 16]).

In [9, 10] the authors of this paper were interested in counting spanning trees and other structures in *undirected circulant* graphs. The main tool introduced there was a recursive decomposition of such graphs. In Section 2 we describe a related recursive decomposition of *directed circulant* graphs. Our technique will be to use this decomposition to show that for some constant m there is a $m \times 1$ (column) vector function $\bar{T}(n)$ such that

$$\forall n \geq 2\bar{s}, \quad T(n) = \beta\,\bar{T}(n) \quad \text{and} \quad \bar{T}(n+1) = A\,\bar{T}(n) \tag{4}$$

where $\bar{s}$ is a constant to be defined later (but reduces to $\bar{s} = s_k$ for the Minc formulation described previously), β is a $1 \times m$ constant row-vector and A is a constant $m \times m$ matrix. Such an A is known as a *transfer-matrix* see, e.g., [19].

Let $P(x) = \sum_{i=0}^{t} p_i x^i$ be any polynomial that annihilates A, i.e., $P(A) = 0$. Then it is easy to see that $\forall n \geq 2\bar{s}$,

$$\sum_{i=0}^{t} p_i T(n+i) \quad = \quad \beta \left(\sum_{i=0}^{t} p_i A^{n+i-2\bar{s}} \right) \bar{T}(2\bar{s})$$

$$
\begin{aligned}
&= \beta\, A^{n-2\bar{s}} \left(\sum_{i=0}^{t} p_i A^i \right) \bar{T}(2\bar{s}) \\
&= \beta\, A^{n-2\bar{s}}\, \mathbf{0}\, \bar{T}(2\bar{s}) \\
&= 0
\end{aligned}
$$

where $\mathbf{0}$ denotes the $m \times m$ zero matrix and 0 a scalar; $T(n)$ thus satisfies the degree-t constant coefficient recurrence relation $T(n+t) = \sum_{i=0}^{t-1} -\frac{p_i}{p_t} T(n+i)$ in n. By the Cayley-Hamilton theorem, the characteristic polynomial of A – which has degree $\leq m$ – must annihilate A, so such a polynomial exists and $T(n)$ satisfies a recurrence relation of at most degree m. In our notation, Minc's theorem is that $T(n)$ satisfies a recurrence relation of degree $2^{\bar{s}} - 1$. Unfortunately, in our construction, $m = 2^{2\bar{s}}$ so the characteristic polynomial does not suffice for our purposes. Our next step will involve showing that even though A is of size $2^{2\bar{s}} \times 2^{2\bar{s}}$, there is a much smaller P, of degree $2^{\bar{s}} - 1$, that annihilates A, thus reproving Minc's theorem. We point out that this degree reduction of the transfer matrix (to less than the square-root of the original size) is, a-priori, quite unexpected, and does not occur in the undirected-circulant counting problems analyzed in [9, 10].

One interesting consequence of this new derivation is that, unlike in Minc's proof, to derive the recurrence relation it is no longer necessary to start by spending $\Omega\left(2^{2^{\bar{s}}}\right)$ time calculating the first $2^{\bar{s}}$ values of $T(n)$ using Ryser's method. Instead one only has to calculate A, β, the polynomial P and the first $2^{\bar{s}}$ values of $\bar{T}(n)$ which, as we will see later, can all be done in $O(\bar{s}2^{4\bar{s}})$ time, reducing the start-up complexity from doubly-exponential in $\bar{s}$ to singularly exponential.

Another, albeit minor, consequence of this new derivation is that it can also handle non-$(0,1)$ circulants. That is, given *any* matrix A_n of the form (1), even when the a_i are not restricted to be in $\{0,1\}$ the technique shows that $\mathrm{Perm}(A_n)$ satisfies a recurrence relation of degree $2^{\bar{s}} - 1$. This is only a minor consequence, though, since working through the details of Minc's original proof it is possible to modify it to get the same result.

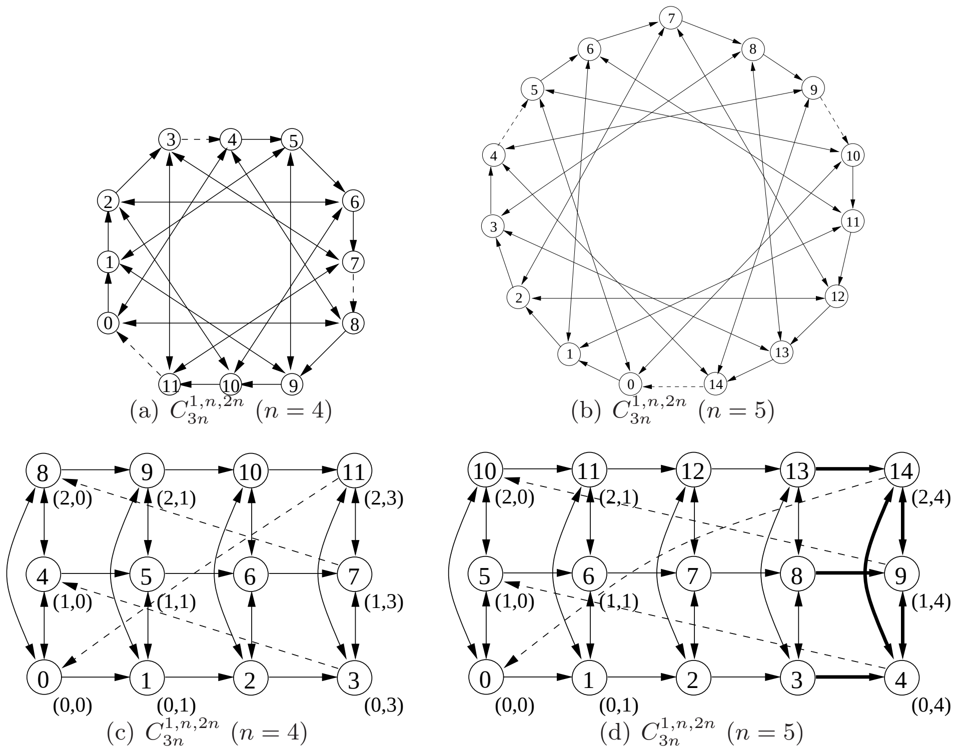
A much more important new consequence, and a major motivation for this paper, is the fact that the proof can be extended to evaluate the permanents of *non-constant (linear)* jump circulant matrices, something which has not been addressed before. As an example Minc's technique would not permit calculating the permanents of $A_{3n}(1, n, 2n)$, something which our new method allows. To explain this, we generalize Definition 2 to

Definition 4 *See Figures 2 (a) and (b). Let p, s, $p_1, p_2, \ldots, p_k$ and $s_1, s_2, \ldots, s_k$ be fixed integral constants with such that $\forall i$, $0 \leq p_i < p$. Set $S = \{p_1 n + s_1, p_2 n + s_2, \cdots, p_k n + s_k\}$. Denote the $(pn+s)$-node* directed circulant graph *with jumps S by*

$$C_n = C_{pn+s}^{p_1 n+s_1, p_2 n+s_2, \cdots, p_k n+s_k} = (V(n), E_C(n))$$

where

$$V(n) = \{0, 1, \ldots, pn+s-1\}$$



(a) $C_{3n}^{1,n,2n}$ $(n=4)$ (b) $C_{3n}^{1,n,2n}$ $(n=5)$

(c) $C_{3n}^{1,n,2n}$ $(n=4)$ (d) $C_{3n}^{1,n,2n}$ $(n=5)$

Figure 2: $C_{3n}^{1,n,2n}$, a non-constant jump circulant: Solid edges are L_n. Dashed edges are $\texttt{Hook}(n)$. (a) and (b) are the circulant graphs when $n = 4, 5$. (c) and (d) are corresponding lattice representations of the same graphs. The bold solid edges on the right of (d) are $\texttt{New}(5) = L_5 - L_4$. The 3 vertices$\{4, 9, 14\}$ on the right are $VN(n)$. Note that the dashed $\texttt{Hook}(n)$ edges for both $n = 4, 5$ are "independent" of n.

and

$$E_C(n) = \Big\{(i, j) : (j - i) \bmod (pn + s) \in S\Big\}.$$

Figure (2a) and (2b) illustrate $C_{3n}^{1,n,2n}$ for $n = 4, 5$. Figure (2c) and (2d) are the corresponding lattice representation, which will be introduced in section 4.

Note that $A_{pn+s}(p_1n + s_1, p_2n + s_2, \cdots, p_kn + s_k)$ is the adjacency matrix of C_n so, counting the cycle-covers in C_n is equivalent to evaluating $\mathrm{Perm}(A_{pn+s}(p_1n+s_1, p_2n+s_2, \cdots, p_kn+s_k))$. Our method of counting the cycle covers in C_n will be to derive a new recursive decomposition of C_n (which might be of independent interest) and use it to show that an analogue of (4) holds in the non-constant jump case as well; thus $T(n)$ still satisfies a constant-coefficient recurrence relation in n. For example, Table 1, shows the recurrence relation for the number of cycle covers in $C_{3n}^{1,n+1,2n}$, $C_{3n}^{0,n,2n-1}$, $C_{3n+1}^{1,n,2n+1}$ and $C_{3n+1}^{2,n+1,2n+2}$.

In the next section we describe the recursive decomposition of C_n, for constant-jump circulants upon which our technique is based. In Section 3 we show how this permits easily reproving Minc's result for constant-jump circu-

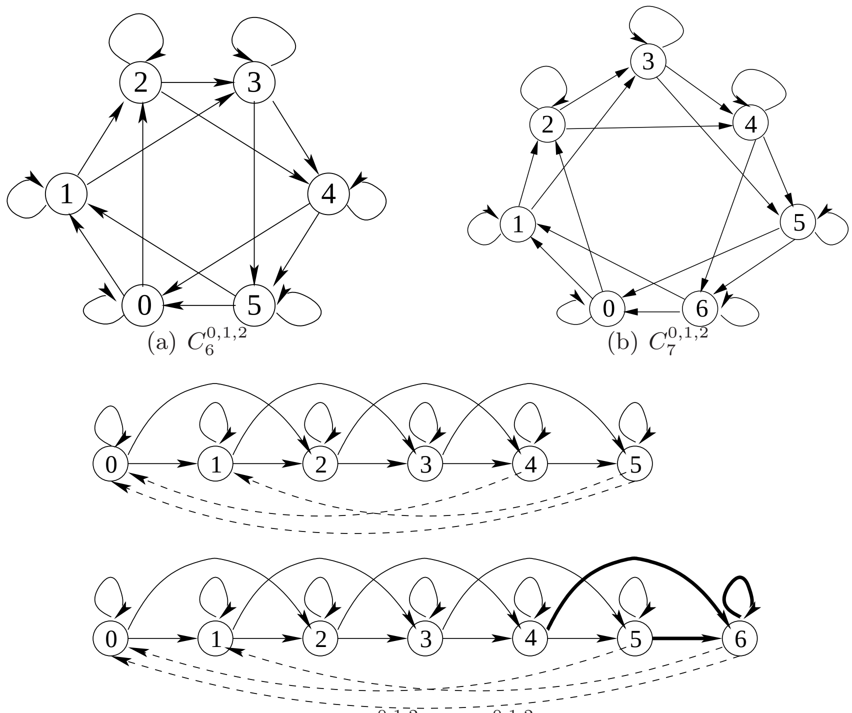
$C_n^{-1,0,1}$ $C_n^{0,1,2}$	$T(n) = 2T(n-1) - T(n-3)$ initial values $9, 13, 12$ for $n = 4, 5, 6$	$T(n) \sim \phi^n$ $\phi = (1+\sqrt{5})/2$
$C_{3n}^{0,n,2n-1}$ $C_{3n}^{1,n+1,2n}$	$T(n) = 5T(n-1) - 5T(n-2)$ $-5T(n-3) + 6T(n-4)$ initial values $17, 45, 113, 309$ for $n = 2, 3, 4, 5$	$T(n) \sim 3^n$
$C_{3n+1}^{1,n,2n+1}$ $C_{3n+1}^{2,n+1,2n+2}$	$T(n) = 4T(n-1) + 5T(n-2)$ $-16T(n-3) - 2T(n-4)$ $-8T(n-5) - 6T(n-6)$ $+16T(n-7) + 3T(n-8)$ $+4T(n-9) + T(n-10)$ initial values $31, 169, 523, 2401, 9351, 40401,$ $167763, 714025, 3010351, 12766329$ for $n = 2, 3, \ldots, 11$	$T(n) \sim \psi\phi^n$ $\psi = (1+\sqrt{5})/2$ $\phi = 2+\sqrt{5}$

Table 1: The number of cycle-covers $T(n)$ in directed circulant graphs with constant jumps $C_n^{-1,0,1}$ and $C_n^{0,1,2}$, and with non-constant jumps $C_{3n}^{0,n,2n-1}$, and $C_{3n}^{1,n+1,2n}$, and $C_{3n+1}^{1,n,2n+1}$ and $C_{3n+1}^{2,n+1,2n+2}$ as derived by the techniques of this paper. Note that for all pairs of graphs, the number of cycle covers for each of the graphs in the pair is the same. This is because the adjacency matrices for the two items in each pair are just linear circular shifts of each other so the permanents of their adjacency matrices are the same. The second item in each pair is in the form that we analyze. That is, for the constant case, having $s_1 = 0$, and for the nonconstant case, having, $\forall i,\ s_i \geq s$.

lants. In Section 4 we then describe the generalization of the decomposition and the minor modifications to the proofs that are needed to extend our analysis to the non-constant circulants introduced in Definition 4. Finally, in Section 5, we sketch generalizations and other uses of our technique; we first show how it can be extended to calculate permanents of non 0-1 circulants. We then describe how it can be used to calculate the moments of the the random variable counting the number of cycles in a random restricted permutations. We conclude by discussing how to extend the technique to counting the number of Hamiltonian cycles in directed circulants, extending the result of [21], which only worked for circulant graphs with two jumps.

2 A Recursive Decomposition of Directed Circulant Graphs

The main conceptual difficulty with deriving a recurrence relation for $T(C_n)$ is that larger circulant graphs can not be built recursively out of smaller ones. The crucial observation, though, is that, there is *another graph*, L_n, the *lattice graph*, that *can* be built recursively, and C_n can then be constructed from L_n



(a) $C_6^{0,1,2}$ (b) $C_7^{0,1,2}$

(c) $L_6^{0,1,2}$ and $L_7^{0,1,2}$

Figure 3: $C_n^{0,1,2}$, a constant jump circulant and its lattice equivalents. In (c) solid edges are L_n; dashed edges are $\texttt{Hook}(n)$; bold edges in $C_7^{0,1,2}$ are $\texttt{New}(n)$ for $n = 7$.

through the addition of a constant number of edges[1]. In [9, 10] the authors of this paper developed such a recursive decomposition for *undirected* circulant graphs as a tool for counting the number of spanning trees in such graphs. In what follows we develop a corresponding decomposition for *directed circulants* that will permit counting cycle-covers.

We first show this for the restricted case in which S, the set of jumps, is constant (independent of n), where it is easy to visualize. In Section 4 we will see how to extend the decomposition to the more complicated case in which the set of jumps can depend linearly upon n, as described in Definition 2.

We assume that $0 = s_1 < s_2 < \cdots < s_k$ and set $\bar{s} = s_k$. Figure (3) shows two circulant graphs with constant jumps 0, 1, 2. Note that our assumption is without loss of generality, as we can choose any row of a circulant matrix to be the top one; for our assumption to be correct, we choose a row with a '1' in its first position. Equivalently, multiplying a circulant matrix by P_n or P_n^{-1} doesn't change its permanent so we can normalize $S_1 = 0$. For example, $P_n^{-2} + P_n^{-1} + I$, $P_n^{-1} + I + P_n$ and $I + P_n + P_n^2$, corresponding respectively, to graphs $C_n^{-2,-1,0}$, $C_n^{-1,0,1}$ and $C_n^{0,1,2}$, all have the same permanent.

[1]To put this into context, this is very similar to the definition of *Recursive families* for undirected graphs [3, 18], which were used for recursively building the Tutte polynomials of graphs in a class.

Definition 5 *See Figure 3. Let* $S = \{s_1, s_2, \ldots s_k\}$*, where the* s_i *are fixed integers. Define the* n*-node* lattice graph[2] *with jumps* S *by*

$$L_n^{s_1,s_2,\ldots,s_k} = (V(n), E_L(n))$$

where

$$E_L(n) = \Big\{(i,j) : j - i \in S\Big\}.$$

Now set

$$\texttt{Hook}(n) = E_C(n) - E_L(n)$$

and

$$\texttt{New}(n) = E_L(n+1) - E_L(n).$$

Note that this implies

$$L_{n+1} = L_n \cup \texttt{New}(n) \quad \textit{and} \quad C_n = L_n \cup \texttt{Hook}(n). \tag{5}$$

The simple but important observation is that, *when* n *is viewed as a label rather than as a number*, $\texttt{Hook}(n)$ and $\texttt{New}(n)$ are *independent* of the actual *value* of n.

Lemma 1

$$\begin{aligned} \texttt{Hook}(n) &= \bigcup_{s\in S} \{\, (n-j,\, s-j) \,:\, 1 \le j \le s \}\,, \\ \texttt{New}(n) &= \bigcup_{s\in S} \{(n-s, n)\}. \end{aligned}$$

Set $\bar{s} = s_k$*. Now define*

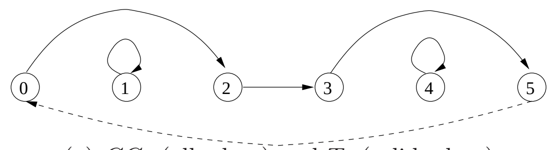
$$\begin{aligned} L(n) &= \{0, \ldots \bar{s} - 1\}, \\ R(n) &= \{n - \bar{s}, \ldots, n-1\}. \end{aligned}$$

Then

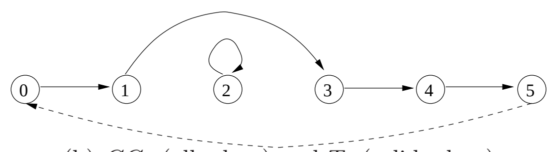
$$\texttt{Hook}(n) \subseteq (R(n) \times L(n)) \tag{6}$$
$$\texttt{New}(n) \subseteq (R(n) \times \{n\}) \cup \{(n,n)\} \tag{7}$$

Important Note: In this section and the next we will always assume that $n \ge 2\bar{s}$ *since this will guarantee that* $L(n) \cap R(n) = \emptyset$*. Without this assumption some of our proofs would fail. Also note that the* $\{(n,n)\}$ *term in* $\texttt{New}(n)$ *appears because* $0 \in S$*.*

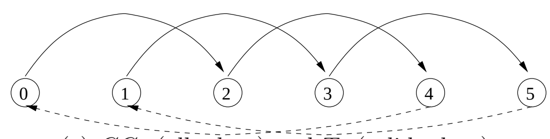
[2]The reason for calling this a *lattice* graph will become visually obvious later in Definition 12, which generalizes this definition to the non-constant jump case.



(a) CC_1 (all edges) and T_1 (solid edges)



(b) CC_2 (all edges) and T_2 (solid edges)



(c) CC_3 (all edges) and T_3 (solid edges)

Figure 4: All of the figures are subsets of $C_6^{0,1,2}$. Solid edges are in L_n; dashed edges are in $\texttt{Hook}(n)$. The solid plus dashed edges comprise three different cycle covers CC_i, $i = 1, 2, 3$ in C_6. Removing the dashed $\texttt{Hook}(n)$ edges leaves three *legal* covers T_i, $i = 1, 2, 3$, in L_6. Note that $\bar{s} = 2$, $C(T_1) = C(T_2) = (\,(0,1),\ (0,1)\,)$ and $C(T_3) = (\,(0,0),\ (0,0)\,)$.

3 A New Proof of Minc's result

Let CC be a cycle-cover of C_n, i.e., $\forall v$, $\texttt{ID}_T(v) = \texttt{OD}_T(v) = 1$. Then, from (3), in $T = CC - \texttt{Hook}(n)$, almost all vertices v except (possibly) some of those that have an edge of $\texttt{Hook}(n)$ hanging off of them, have $\texttt{ID}_T(v) = \texttt{OD}_T(v) = 1$. This motivates

Definition 6 *$T \subseteq E_L(n)$ is a* legal cover *of L_n if*

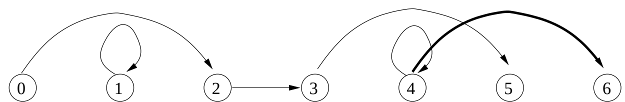
- $\forall v \in V, \quad \texttt{ID}_T(v) \le 1$ *and* $\texttt{OD}_T(v) \le 1$.
- $\forall v \in V - L(n), \quad \texttt{ID}_T(v) = 1$.
- $\forall v \in V - R(n), \quad \texttt{OD}_T(v) = 1$.

Then, from (5) we have

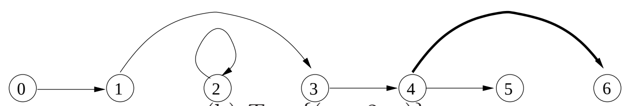
Lemma 2
(a) If $T \subseteq E_C(n)$ is a cycle-cover of C_n, then
$\quad$ *$T - \texttt{Hook}(n)$ is a legal-cover of L_n.*

(b) If $T \subseteq E_L(n+1)$ is a legal-cover of L_{n+1}, then
$\quad$ *$T - \texttt{New}(n)$ is a legal-cover of L_n.*

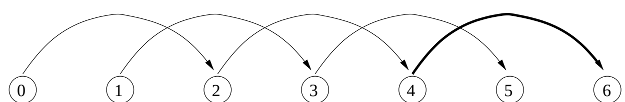
From the definition of legal covers we can classify and partition legal covers by the appropriate in/out degrees of their vertices in $L(n), R(n)$.



(a) $T_1 \cup \{(n-2, n)\}$



(b) $T_2 \cup \{(n-2, n)\}$



(c) $T_3 \cup \{(n-2, n)\}$

Figure 5: n was increased from 6 to 7 and $S = \{(4,6)\} \subseteq \text{New}(6)$ was added to the T_i of the previous figure. Note that, in L_7, $C(T_1 \cup S) = C(T_2 \cup S) = \emptyset$ since they are no longer legal covers. Also, $C(T_3 \cup S) = (\,(0,0),\ (0,0)\,)$.

Definition 7 *A is a* binary r-tuple *if*

$$A = (A(0), A(1), \ldots, A(r-1)) \text{ where } \forall i, A(i) \in \{0,1\}.$$

Definition 8 *(See Figure 4). Let $\mathcal{P}$ be the set of $2^{2\bar{s}}$ possible binary tuple pairs (L, R) where each of L, R are, respectively, binary $\bar{s}$ tuples.*

Let T be a legal-cover *of L_n. The* classification of T *will be* $C(T) = (L^T, R^T) \in \mathcal{P}$ *where*

$$\begin{aligned} \forall 0 \le i < \bar{s}, \quad L^T(i) &= \mathtt{ID}_T(i) \\ R^T(i) &= \mathtt{OD}_T(n-1-i). \end{aligned}$$

If T is not *a legal-cover then we will write $C(T) = \emptyset$. Finally, set*

$$\begin{aligned} \mathcal{L}(n) &= \{T \subseteq E_L(n) : T \text{ is a legal cover of } L_n\} \\ \mathcal{L}_X(n) &= \{T \in \mathcal{L}(n) : C(T) = X\} \\ T_X(n) &= |\mathcal{L}_X(n)| \end{aligned}$$

so $T_X(n)$ is the number of legal-covers of L_n with classification X.

The main reason for introducing these definitions is that checking whether a legal cover T of L_n can be completed to a cycle-cover of C_n or to a legal cover in L_{n+1} doesn't depend upon all of T but *only upon its classification $C(T)$*.

Lemma 3 *See Figures 4 and 5.*
Let $X = (L^X, R^X) \in \mathcal{P}$. Let T_1 be a legal cover in L_{n_1} and T_2 be a legal cover of L_{n_2}, such that $C(T_1) = C(T_2) = X$.

(a) Let $S \subseteq \mathtt{Hook}(n)$. Then,

$$T_1 \cup S \text{ is a cycle-cover of } C_{n_1}$$

if and only if
$T_2 \cup S$ is a cycle-cover of C_{n_2}

(b) Let $S \subseteq \texttt{New}(n)$. Then,

$$C(T_1 \cup S) = C(T_2 \cup S).$$

That is, either both *$T_1 \cup S$ and $T_2 \cup S$ are* not *legal covers or, they are both legal covers and there is some $X' \in \mathcal{P}$ such that $C(T_1 \cup S) = C(T_2 \cup S) = X'$*

Proof. To prove (a) recall that $T \cup S$ is a cycle-cover of L_n if and only if, $\forall v \in V$, $\texttt{ID}_{T\cup S}(v) = \texttt{OD}_{T\cup S}(v) = 1$ or

$$\forall v \in V, \texttt{ID}_S(v) = 1 - \texttt{ID}_T(v) \quad \text{and} \quad \texttt{OD}_S(v) = 1 - \texttt{OD}_T(v) \tag{8}$$

From Lemma 1 and the definition of a legal cover we have that this is true if and only if

$$\begin{array}{rrcl} \forall i < \bar{s}, & \texttt{ID}_S(i) & = & 1 - L^X(i), \\ & \texttt{OD}_S(n-1-i) & = & 1 - R^X(i). \end{array}$$

and this is only dependent upon X and S and not upon n or any other properties of T.

The proof of (b) is similar and omitted here.

□

This lemma permits us, for $X, X' \in \mathcal{P}$ and $S \subseteq \texttt{Hook}(n)$, to abuse the notations and write $(X \cup S) = X'$ to denote that, when $C(T) = X$, $C(T \cup S) = X'$. We will sometimes also write "$X \cup S$ *is a cycle cover*" to denote that $T \cup S$ is a cycle cover.

Definition 9 *For $X, X' \in \mathcal{P}$, $S \subseteq \texttt{Hook}(n)$ and $S' \subseteq \texttt{New}(n)$ set*

$$\beta_{X,S} = \begin{cases} 1 & \text{if } X \cup S \text{ is a cycle cover} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\alpha_{X,X',S'} = \begin{cases} 1 & \text{if } C(X' \cup S') = X \\ 0 & \text{otherwise} \end{cases}.$$

Now set

$$\beta_X = \sum_{S \subseteq \texttt{Hook}(n)} \beta_{X,S}$$

and

$$\alpha_{X,X'} = \sum_{S' \subseteq \texttt{New}(n)} \alpha_{X,X',S'}. \tag{9}$$

Note that β_X and $\alpha_{X,X'}$ are constants that can be mechanically calculated. In fact $\alpha_{X,X'}$ is much simpler to calculate than it might initially appear seem since

Lemma 4 *If* $\alpha_{X,X',S'} = 1$, *then* $|S'| = 1$.

Proof. In order for $X' \cup S$ to be a legal cover S must include at least one edge that points to vertex n, so $|S| \geq 1$. From (7), *all* edges in $\texttt{New}(n)$ point to n. If $|S'| > 1$, then $\texttt{ID}_{X' \cup S}(n) = |S'| > 1$ and $X' \cup S$ wouldn't be a legal cover. $\square$

Thus (9) can be calculated by summing over $|\texttt{New}(n)| = k$ values, instead of 2^k values.

Lemmas 2 and 3 immediately imply our first technical result.

Lemma 5

$$T(n) = \sum_{X \in \mathcal{P}} \beta_X T_X(n)$$

and

$$T_X(n+1) = \sum_{X' \in \mathcal{P}} \alpha_{X,X'} T_X(n).$$

Let $m = |\mathcal{P}| = 2^{2\bar{s}}$. Take any arbitrary ordering of $\mathcal{P}$ and define the $1 \times m$ constant vector $\beta = (\beta_X)_{X \in \mathcal{P}}$ and $m \times m$ constant matrix $A = (\alpha_{X,X'})_{X,X' \in \mathcal{P}}$. Finally, set $\bar{T}(n) = \mathrm{col}(T_X(n))_{X \in \mathcal{P}}$ to be a $m \times 1$ column vector. Then, Lemma 5 is exactly

$$\forall n \geq 2\bar{s}, \qquad T(n) = \beta \bar{T}(n) \quad \text{and} \quad \bar{T}(n+1) = A\bar{T}(n)$$

which is equation (4). As mentioned in the introduction, this immediately implies that $T(n)$ satisfies a fixed-degree constant coefficient recurrence relation where the degree of the recurrence is at most the degree of any polynomial $P(x)$ such that $P(A) = 0$. By the Cayley-Hamilton theorem, $Q(A) = 0$, $Q(x)$ is the degree $m = 2^{2\bar{s}}$ characteristic polynomial $Q(x) = \det(IX - A)$.

We will now see that it is possible to reduce this degree from $2^{2\bar{s}}$ down to below $2^{\bar{s}}$. We will do this by showing that, given appropriate orderings of the classifications, $A = (\alpha_{X,X'})$ will have a very special block diagonal format. In what follows, please refer to the worked example in Appendix A for illustration.

Definition 10 *A linear ordering on the classifications* $\mathcal{P}$ *will be called* consistent *if it is the lexicographic concatenation of linear orderings on its left and right components.*

More specifically, linear ordering "$<$" is consistent on $\mathcal{P}$ *if there exist linear orderings "$\leq_L$" and "$\leq_R$" such that if* $X_1 = (L_1, R_1)$ *and* $X_2 = (L_2, R_2)$ *we have* $X_1 < X_2$ *if and only if one of the following is true*

$$L_1 <_L L_2 \qquad \text{or} \qquad L_1 =_L L_2 \;\; \text{and} \;\; R_1 <_R R_2$$

Note that in the above definition it is not necessary for the ordering on the left component to be the same as the ordering on the right one (we will use this fact later in Lemma 7).

Lemma 6 *Let $A = (\alpha_{X,X'})$. If $X \in \mathcal{P}$ is ordered consistently, then*

$$A = \begin{pmatrix} \bar{A} & 0 & \cdots & 0 \\ 0 & \bar{A} & 0 & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{A} \end{pmatrix} \tag{10}$$

where $\bar{A}$ is some $2^{\bar{s}} \times 2^{\bar{s}}$ matrix. That is, $A = diag(\bar{A}, \bar{A}, \ldots, \bar{A})$ where A contains $2^{\bar{s}}$ copies of $\bar{A}$ on its diagonal.

Proof. Suppose $X = (L^X, R^X)$ and $X' = (L^{X'}, R^{X'})$.

Recall that $\alpha_{X,X'} = \sum_{S \subseteq \texttt{New}(n)} \alpha_{X,X',S}$ where $\alpha_{X,X',S} = 1$ if and only if $C(X' \cup S) = X$, and is otherwise 0.

Let L denote any binary $\bar{s}$-tuple. Partition $\mathcal{P}$ up into $2^{\bar{s}}$ sets of size $2^{\bar{s}}$, $\mathcal{P}_L = \{X \in \mathcal{P} : L^X = L\}$.

Note that, from Lemma 1, if $S \subseteq \texttt{New}(n)$, none of S's edges have endpoints in $L(n)$. Intuitively, this is because edges in $\texttt{New}(n)$ only connect vertices near the *right* side of the lattice and do not touch any vertices on the *left* side of the lattice.

Thus, if $\alpha_{X,X',S} = 1$, then $L^X = L^{X'}$. In particular this means that if $\alpha_{X,X',S} = 1$ then X, X' are both in the same partition set $\mathcal{P}_L$.

Now suppose that $\alpha_{X,X',S} = 1$. Let $\bar{L}$ be *any* other binary $\bar{s}$-tuple and set

$$\bar{X} = (\bar{L},\, R^X) \quad \text{and} \quad \bar{X}' = (\bar{L},\, R^{X'}). \tag{11}$$

Then, again using the fact that none of the endpoints of S are in $L(n)$ we have that $C(X' \cup S) = X$ if and only if $C(\bar{X}' \cup S) = \bar{X}$ so $\alpha_{X,X'} = \alpha_{\bar{X},\bar{X}'}$.

When constructing matrix $A = (\alpha_{X,X'})_{X,X' \in \mathcal{P}}$ we previously allowed any arbitrary ordering of $\mathcal{P}$. Ordering the $X \in \mathcal{P}$ consistently groups all of the X in a particular $\mathcal{P}_L$ consecutively. The observations above imply that A is partitioned into $2^{\bar{s}} \times 2^{\bar{s}}$ blocks where each block is of size $2^{\bar{s}} \times 2^{\bar{s}}$. The non-diagonal blocks correspond to $\alpha_{X,X'}$ where X, X' are in different partitions so all of the non-diagonal blocks are 0. On the other hand, the fact that $\alpha_{X,X'} = \alpha_{\bar{X},\bar{X}'}$ for the $\bar{X}, \bar{X}'$ defined in (11) and the consistency of the ordering of the X tells us that all the diagonal blocks are copies of each other, i.e., we have proven (10). □

Corollary 1 *There is a degree $2^{\bar{s}}$ polynomial $P(x)$ such that $P(A) = 0$.*

Proof. From the previous lemma, any polynomial $P(x)$ that annihilates $\bar{A}$ also annihilates A. Since $\bar{A}$ is a $2^{\bar{s}} \times 2^{\bar{s}}$ matrix, the Cayley-Hamilton theorem says that the characteristic polynomial $\bar{P}(x)$ of $\bar{A}$, which is of degree $2^{\bar{s}}$, annihilates $\bar{A}$. □

The original Minc result [14, 17]) gave an order of $2^{\bar{s}} - 1$. We can derive this through a slightly more sophisticated decomposition of $\bar{A}$.

Lemma 7 *Let $A = (\alpha_{X,X'})$. Then there is a degree $2^{\bar{s}} - 1$ polynomial $P(x)$ such that $P(A) = 0$.*

Proof. If $\alpha_{X,X',S} = 1$, then we have just seen that $L^X = L^{X'}$. Consider R^X, $R^{X'}$ and S.

First recall from Lemma 4 that if $\alpha_{X,X',S} = 1$ then S contains exactly one edge.

We claim that if $\alpha_{X,X',S} = 1$ then X and X' must contain exactly the same number of '0's. Note that since $L^X = L^{X'}$ we only need to show that R^X and $R^{X'}$ have the same number of '0's.

There are actually two cases. The first case is that $S = \{(n-s, n)\}$. In this case we are throwing away one vertex $(n-s)$ which (because of the legality of X') had outdegree zero and adding a new vertex n which also has outdegree 0. So, the number of '0's in $X \cup S$ is the same as the number of '0's in X'.

The second case is that $S = \{(n-i, n)\}$ where $i < s$.. Since $X' \cup S$ is legal, vertex $n - s$ must have already had outdegree one so throwing it away doesn't change the number of '0's. Adding the new vertex n with outdegree '0' increases the number of '0's by one. Adding edge $(n-i, n)$ changes the outdegree of vertex $n - i$ to one, decreasing the number of '0's by one.

So, the number of '0's in $X' \cup S$ is again the same as the number of '0's in X.

Recall that we have that $L^X = L^{X'}$. This suggests that we can re-order the entries of $\bar{A}$ so that all R^X with the same number of 0's are grouped together (maintaining the fact that the ordering is consistent). Since there are $\binom{\bar{s}}{i}$ $\bar{s}$-tuples containing i '0's, $\bar{A}$ will become a block diagonal matrix of $s+1$ blocks with block i having size $\binom{\bar{s}}{i}$. That is

$$\bar{A} = \begin{pmatrix} B_0 & 0 & \cdots & 0 & 0 \\ 0 & B_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & B_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & B_n \end{pmatrix} \tag{12}$$

where B_i is a $\binom{\bar{s}}{i} \times \binom{\bar{s}}{i}$ matrix..

Let $P_i(x)$ be the characteristic polynomial of B_i. This has degree $\le \binom{\bar{s}}{i}$.

Note that B_0 and B_n are both 1×1 matrices. By construction, $B_0 = B_n = (1)$ so $P_0(x) = P_n(x) = 1 - x$, i.e., their characteristic polynomial is the same..

Because of the block diagonal form of B, $P(x) = \prod_{i=0}^{n-1} P_i(x)$ annihilates B. This polynomial has degree $\le \sum_{i=0}^{n-1} \binom{\bar{s}}{i} = 2^{\bar{s}} - 1$, proving the lemma. □

Lemma 5 tells us that (4) holds while Lemmas 6 and 7 tell us that matrix A is annihilated by polynomial $P(x)$ of degree $2^{\bar{s}} - 1$. Combining them gives that $T(n)$ satisfies a degree-$(2^{\bar{s}} - 1)$ constant coefficient recurrence relation.

3.1 Deriving the Recurrence Relation

We have just seen that $T(n)$ *satisfies* a degree-$(2^{\bar{s}} - 1)$ constant coefficient recurrence relation. To actually *derive* the recurrence relation we must construct

- (i) a polynomial $Q(n)$ that annihilates $A = \{\alpha_{X,X'}\}$ and
- (ii) the initial conditions $T(n)$, $n = 2\bar{s}$, $2\bar{s}+1$, $\ldots$, $2\bar{s}+2^{\bar{s}}-2$.

To construct $Q(x)$, note from Lemma 6 that it suffices to calculate the characteristic polynomials $Q(x)$ of matrix $\bar{A}$. We must therefore first calculate the $2^{2\bar{s}}$ $\alpha_{X,X'}$ entries of $\bar{A}$.

Recall that $\alpha_{X,X'} = \sum_{S \subseteq \mathtt{New}(n)} \alpha_{X,X',S}$ and, as noted in the proof of Lemma 7, we know that if $\alpha_{X,X',S'} = 1$, then S contains at most one edge. Since $\mathtt{New}(n)$ contains $\bar{s}$ edges, we can, with the appropriate data structures, calculate $\alpha_{X,X'}$ in $O(\bar{s}^2)$ time. We can therefore calculate all the non-zero entries in the $\bar{A}$ in $O(\bar{s}^2 2^{2\bar{s}})$ time. Finally, we can calculate $Q(x)$ in $O(2^{3\bar{s}})$ time, since it takes $O(n^3)$ time to compute the characteristic polynomial of an $n \times n$ matrix [11]. Thus, we can calculate $Q(x)$ in $O(2^{3\bar{s}})$ time.

To derive (ii), the initial conditions $T(n)$, $n = 2\bar{s}$, $2\bar{s}+1$, $\ldots$, $2\bar{s}+2^{\bar{s}}-2$. suppose first that we already knew $\bar{T}(2\bar{s})$ and β. Since $\bar{T}(n+1) = A\bar{T}(n)$ we can use the block structure from (10) to calculate $\bar{T}(n+1)$ from $\bar{T}(n)$ in $O(2^{3\bar{s}})$ time. It then takes only another $2^{2\bar{s}}$ time to calculate $T(n+1) = \beta\bar{T}(n+1)$. So, we can calculate all of the values $T(n)$, $n = 2^{\bar{s}}+1, 2^{\bar{s}}+1, \ldots, 2\bar{s}+2^{\bar{s}}-2$ in $O(2^{4\bar{s}})$ time, improving upon the doubly exponential procedure implied by Minc's original result.

It still remains to calculate $T_X(2\bar{s})$ and β_X for all classifications X.

Let $X = (L, R)$. We want to calculate the number of legal covers in $L_{2\bar{s}}$ with classification X. In a legal cover the number of '0's in L must be equal to the number of '0's in R. Let $a_1, a_2, \ldots, a_i$ be the indices such that $L(i) = 0$ and $b_1, b_2, \ldots, b_i$ be the indices such that $R(i) = 0$. Define the set of i edges $A = \bigcup_{j=1}^{i}\{(2\bar{s}-1-b_j, a_j)\}$. Now define a new graph G_X as follows: (a) start with the lattice graph $L_{2\bar{s}}$; (b) remove all edges *entering* vertices a_j, $j = 1, 2, \ldots, i$; (c) remove all edges *leaving* vertices $2\bar{s}-1-b_j$, $j = 1, 2, \ldots, i$; (d) add the i edges in A. Then it is not difficult to see that T is a legal cover in $L_{2\bar{s}}$ if and only if $T \cup A$ is a cycle cover of G_X. Since every cycle cover of G_X *must* contain all edges in A there is a one-one correspondence between cycle covers in G_X and legal covers in $L_{2\bar{s}}$ with classification X. We can therefore calculate $T_X(2\bar{s})$ by calculating the permanent of the adjacency matrix of G_X which can be done in $O(s2^{2\bar{s}})$ time using Ryser's algorithm. Calculating all entries in $\bar{T}(2\bar{s})$ then takes $O(s2^{4\bar{s}})$ time.

Finally, we must calculate all the β_X. Let $X = (L, R)$. If $\beta_X \neq 0$ then the number of '0's in L must be equal to the number of '0's in R. As above, let $a_1, a_2, \ldots, a_i$ be the indices such that $L(i) = 0$ and $b_1, b_2, \ldots, b_i$ be the indices such that $R(i) = 0$. Now construct the $i \times i$ bipartite graph B as follows:

Edge $(j, k) \in B$ if and only if $(n-1-b_j, a_k) \in \mathtt{Hook}(2\bar{s})$.

It is not difficult to see that β_x is exactly the number of complete matchings in B. We can therefore calculate β_X by evaluating the permanent of the adjacency matrix of B. This can be done in $O(\bar{s}2^{\bar{s}})$ time per entry and thus in $O(\bar{s}2^{3\bar{s}})$ time in total.

Combining everything, we see that we can construct the recurrence relation and initial conditions using $O(s2^{4\bar{s}})$ time.

4 Non-constant Jump Circulant Graphs

We now extend the above definitions and lemmas to the case of non-constant circulants $C_n = C_{pn+s}^{p_1 n+s_1,\, p_2 n+s_2,\, \cdots,\, p_k n+s_k}$ as introduced in Definition 4. Note that if $s = \alpha p + \beta$ for some arbitrary integer α and integer $\beta \geq 0$, we can rewrite C_n as

$$C_{p(n+\alpha)+\beta}^{p_1(n+\alpha)+s_1',\, p_2(n+\alpha)+s_2',\, \cdots,\, p_k(n+\alpha)+s_k'}$$

where $\forall i, s_i' = s_i - \alpha p$. Thus, we may assume $0 \leq s < p$.

Note that, using a similar argument to that in the previous section preceding Definition 5, we may and do, without loss of generality, assume $\forall i,\ s_i \geq s$.

Analyzing non-constant jump circulants will require a change in the way that we visualize the nodes of C_n; until, now, as in Figure 1(c), we visualized them as points on a line with the edges in $\texttt{Hook}(n)$ connecting the left and right endpoints of the line. In the non-constant jump case it will be convenient to visualize them as points on a bounded-height *lattice*, where $\texttt{Hook}(n)$ connects the left and right boundaries of the lattice. We start by introducing a new graph:

Definition 11 *See Figures 2 and 6. Let $p, s,\ p_1, p_2, \ldots, p_k$ and $s_1, s_2, \ldots, s_k$ be given non-negative integral constants such that $\forall i,\ 0 \leq p_i < p$ and $s_i \geq s \geq 0$. Set $S = \{p_1 n+s_1, p_2 n+s_2, \cdots, p_k n+s_k\}$. For u, v and integer n, set $f(n; u, v) = un + v$. Define*

$$\widehat{C}_n = \left(\widehat{V}_C(n),\ \widehat{E}_C(n)\right)$$

where

$$\widehat{V}(n) = \left\{(u,v) \,\middle|\, \begin{array}{l} 0 \leq u \leq p-2 \\ 0 \leq v \leq n-1 \end{array}\right\} \bigcup \{\, (p-1, v)\ :\ 0 \leq v \leq n+s-1\}$$

and

$$\widehat{E}_C(n) = \left\{ ((u_1,v_1),(u_2,v_2)) \,\middle|\, \begin{array}{l} (u_1,v_1),(u_2,v_2) \in \widehat{V}_C(n) \text{ and} \\ \quad \exists i \text{ such that} \\ \quad f(n;u_2,v_2) - f(n;u_1,v_1) = p_i n + s_i \bmod (pn+s) \end{array} \right\}$$

If $s = 0$ we see that $\widehat{C}_n$ is simply a rectangular lattice with other regularly placed edges as in Figure 2. If $s > 0$ then $\widehat{C}_n$ is a rectangular lattice with extra vertices extending out from its top row as in Figure 6. These extra vertices, which disturb the regularity of the lattice, are what will complicate our analysis.

Directly from the definition we see $\widehat{C}_n$ is *isomorphic* to $C_n = C_{pn+s}^{p_1 n+s_1, p_2 n+s_2, \cdots, p_k n+s_k}$. In particular, cycle-covers of $\widehat{C}_n$ are in 1-1 correspondence with cycle covers of C_n so we can restrict ourselves to counting cycle covers of $\widehat{C}_n$. We now introduce the generalization of Definition 5.

Definition 12 *Let $p, s,\ p_1, p_2, \ldots, p_k$ and $s_1, s_2, \ldots, s_k$ and S, f be as in Definition 11. Define the $pn+s$-node* lattice graph *with jumps S*

$$L_n = \left(\widehat{V}(n), \widehat{E}_L(n)\right)$$

where

$$\widehat{E}_L(n) = \left\{ ((u_1,v_1),(u_2,v_2)) \;\middle|\; \begin{array}{l} (u_1,v_1),(u_2,v_2) \in \widehat{V}_C(n) \textit{ and} \\ \quad \exists i \textit{ such that} \\ \quad (a)\, f(n;u_2,v_2) - f(n;u_1,v_1) = p_i n + s_i \bmod (pn+s) \\ \quad \textbf{and} \\ \quad (b)\, u_2 - u_1 = p_i \bmod p \end{array} \right\}$$

Now set

$$\texttt{Hook}(n) = \widehat{E}_C(n) - \widehat{E}_L(n)$$

and

$$\texttt{New}(n) = \widehat{E}_L(n+1) - \widehat{E}_L(n).$$

Note that this implies

$$\begin{array}{rcl} L_{n+1} & = & L_n \cup \texttt{New}(n) \\ & \textit{and} & \\ \widehat{C}_n & = & L_n \cup \texttt{Hook}(n). \end{array} \tag{13}$$

We need the following intuitive lemma that was used implicitly in the constant jump case (but was so obvious there that it was not explicitly mentioned). The proof is straightforward but tedious and has therefore been moved to Appendix B.

Lemma 8 *Let* $(u_1,v_1),(u_2,v_2) \in \widehat{V}(n)$, *and* $e = ((u_1,v_1),(u_2,v_2))$,

$$e \in \widehat{E}_L(n) \Leftrightarrow e \in \widehat{E}_L(n+1).$$

In the constant jump case, we were able to define $L(n)$, $R(n)$ such that all edges in $\texttt{Hook}(n)$ went from $R(n)$ to $L(n)$. It turns out that this property remains in the non-constant jump case as well. However, as will be seen from the internals of the proof of Lemma 9, this property is a result of our assumption that $s_i \geq s$ for all i's. If this assumption did not hold, then some $\texttt{Hook}(n)$ edges might go from $L(n)$ to $R(n)$.

It is now straightforward to derive an analogue to Lemma 1 showing that $\texttt{Hook}(n)$ and $\texttt{New}(n)$ are *independent* of the actual *value* of n. Before doing so we will need one more definition:

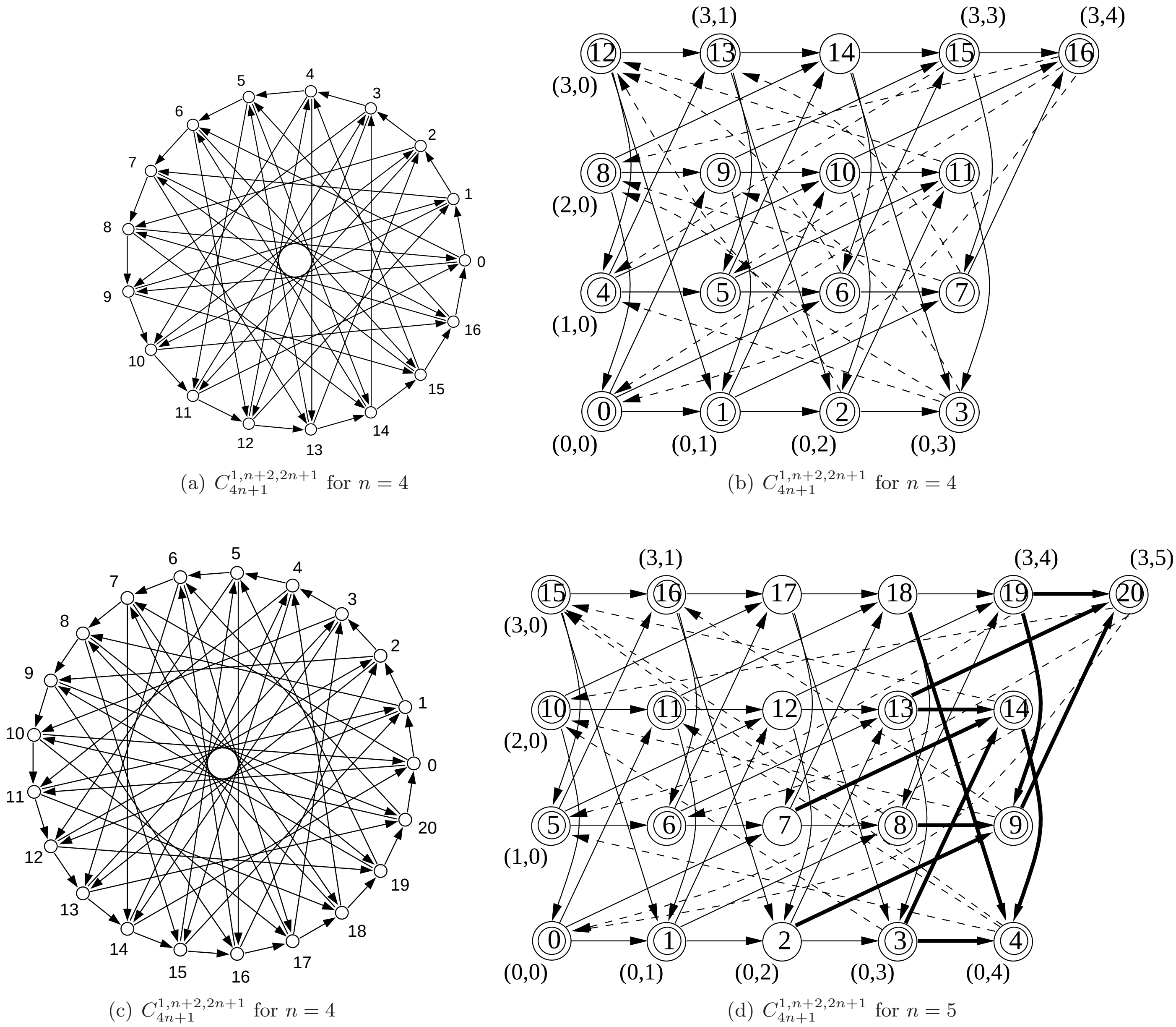
Definition 13

$$NV(n) = V_L(n+1) - V_L(n).$$

$NV(n)$ will be the *new* vertices in $V_L(n+1)$. Note that we did not explicitly define this for *fixed-jump* circulant graphs since in the fixed-jump case $NV(n) = V_L(n+1) - V_L(n) = \{n\}$, i.e., there was only the one new vertex at each step.

Lemma 9 *Set* $\bar{s} = s_k$, *and define*

$$\begin{array}{rcl} L(n) & = & \{(u,v) : 0 \leq u \leq p-1 \text{ and } 0 \leq v \leq \bar{s}-1\} \\ R(n) & = & \{(u,v) : 0 \leq u \leq p-2 \text{ and } n-\bar{s} \leq v \leq n-1\} \\ & & \cup\ \{(p-1,v) : n+s-\bar{s} \leq v \leq n+s-1\} \end{array}$$



(a) $C_{4n+1}^{1,n+2,2n+1}$ for $n = 4$

(b) $C_{4n+1}^{1,n+2,2n+1}$ for $n = 4$

(c) $C_{4n+1}^{1,n+2,2n+1}$ for $n = 4$

(d) $C_{4n+1}^{1,n+2,2n+1}$ for $n = 5$

Figure 6: The graphs $C_{4n+1}^{1,n+2,2n+1}$. (a) and (b) are two representations of $n = 4$; (c) and (d) are $n = 5$. (a) and (c) are drawn in traditional circulant format; (b) and (d) in lattice graph format. The bold edges in (d) are $\texttt{New}(n)$. In the lattice graph format the dashed edges are $\texttt{Hook}(n)$ and the double-circled nodes denote $L(n)$ on the left and $R(n)$ on the right. In (b), $L(n)$ and $R(n)$ actually abut each other; $L(n)$ are the double-circled nodes in the two leftmost columns; $R(n)$ are the double-circled nodes in the three rightmost columns. As discussed in the text, all edges in $\texttt{Hook}(n)$ go from $R(n)$ to $L(n)$.

Then

$$\begin{array}{rcl} \texttt{Hook}(n) & \subseteq & R(n) \times L(n) \\ \texttt{New}(n) & \subseteq & (R(n) \times NV(n)) \cup (NV(n) \times NV(n)) \,. \end{array}$$

The proof is straightforward but tedious and has therefore also been moved to Appendix B. Figures 2 and 6 illustrate the lemma.

In Section 3 we described how to calculate the number of cycle-covers in constant-jump circulant graphs. Reviewing the proof, everything there followed directly as a consequence from the recursive decomposition of circulant graphs in (5) combined with the structural properties of the decomposition given in Lemma 1. But, as we have just seen, non-constant jump circulants and their decompositions have exactly the *same* structural properties, given in (13) and Lemma 9. Therefore, the entire proof developed in Section 3 can be rewritten to work for non-constant jump circulants. The equivalent definitions and lemmas needed in the non-constant jump case are stated below.

Definition 14 $T \subseteq \widehat{E}_L(n)$ *is a* legal cover of L_n *if*

- $\forall v \in V, \quad \texttt{ID}_T(v) \le 1$ *and* $\texttt{OD}_T(v) \le 1$.
- $\forall v \in V - L(n), \quad \texttt{ID}_T(v) = 1$.
- $\forall v \in V - R(n), \quad \texttt{OD}_T(v) = 1$.

Lemma 10
(a) If $T \subseteq \widehat{E}_C(n)$ *is a cycle-cover of* C_n*, then*
$T - \texttt{Hook}(n)$ *is a legal-cover of* L_n.
(b) If $T \subseteq \widehat{E}_L(n+1)$ *is a legal-cover of* L_{n+1}*, then*
$T - \texttt{New}(n)$ *is a legal-cover of* L_n.

The only major rewriting is required in the analogue to Definition 8. The more complicated structure of the lattice graph in the non-constant jump case requires a more complicated function to map the indices of the $L(n)$ and $R(n)$ nodes.

Definition 15 A *is a* binary r-tuple *if*
$A = (A(0), A(1), \ldots, A(r-1))$ *where* $\forall i, A(i) \in \{0, 1\}$. *Let* $\mathcal{P}$ *be the set of* $2^{2p\bar{s}}$ *tuples* (L, R) *where* L, R *are two binary* $p\bar{s}$ *tuples. Let* T *be a* legal-cover *of* L_n. *The* classification of T *will be* $C(T) = (L^T, R^T) \in \mathcal{P}$ *where*

$$\begin{array}{rrcl} \forall 0 \le i < p\bar{s}, & L^T(i) & = & \texttt{ID}_T(g^L(i)) \\ & R^T(i) & = & \texttt{OD}_T(g^R(i)) \end{array}$$

where

$$\begin{array}{rcl} g^L(i) & = & (\lfloor i/\bar{s} \rfloor, i \bmod \bar{s}) \\ g^R(i) & = & \left\{ \begin{array}{ll} (\lfloor i/\bar{s} \rfloor, n-1-(i \bmod \bar{s})) & \lfloor i/\bar{s} \rfloor < p-1 \\ (\lfloor i/\bar{s} \rfloor, n+s-1-(i \bmod \bar{s})) & \textit{otherwise} \end{array} \right. \end{array}$$

Note: g^L and g^R are simply mappings of the indices of the $L^T(i)$ and $R^T(i)$ tuples to the nodes in $L(n)$ and $R(n)$.
If T is not *a legal-cover then we will use the convention that $C(T) = \emptyset$. Finally, set*

$$\begin{aligned}
\mathcal{L}(n) &= \{T \subseteq E_L(n) : T \textit{ is a legal cover of } L_n\} \\
\mathcal{L}_X(n) &= \{T \in \mathcal{L}(n) : C(T) = X\} \\
T_X(n) &= |\mathcal{L}_X(n)|
\end{aligned}$$

so $T_X(n)$ is the number of legal-covers of L_n with classification X.

Lemma 11 *See Figures 7 to 8.*
Let $X = (L^X, R^X) \in \mathcal{P}$. Let T_1 be a legal cover in L_{n_1} and T_2 be a legal cover of L_{n_2}, such that $C(T_1) = C(T_2) = X$.
(a) Let $S \subseteq \texttt{Hook}(n)$. Then,

$$T_1 \cup S \textit{ is a cycle-cover of } C_{n_1}$$
$$\textbf{iff}$$
$$T_2 \cup S \textit{ is a cycle-cover of } C_{n_2}$$

(b) Let $S \subseteq \texttt{New}(n)$. Then,

$$C(T_1 \cup S) = C(T_2 \cup S).$$

That is, either both *$T_1 \cup S$ and $T_2 \cup S$ are* not *legal covers or, they are both legal covers and there is some $X' \in \mathcal{P}$ such that $C(T_1 \cup S) = C(T_2 \cup S) = X'$*

Definition 16 *For $X, X' \in \mathcal{P}$, $S \subseteq \texttt{Hook}(n)$ and $S' \subseteq \texttt{New}(n)$ set*

$$\beta_{X,S} = \begin{cases} 1 & \textit{if } X \cup S \textit{ is a cycle cover} \\ 0 & \textit{otherwise} \end{cases} \quad \text{and} \quad \alpha_{X,X',S'} = \begin{cases} 1 & \textit{if } C(X' \cup S') = X \\ 0 & \textit{otherwise} \end{cases}.$$

Now set

$$\beta_X = \sum_{S \subseteq \texttt{Hook}(n)} \beta_{X,S} \qquad \text{and} \qquad \alpha_{X,X'} = \sum_{S' \subseteq \texttt{New}(n)} \alpha_{X,X',S'}.$$

Because $NV(n)$ is no longer just the one vertex set $\{n\}$, Lemma 4 has to be replaced by

Lemma 12 *If $\alpha_{X,X',S'} = 1$, then $|S'| = |NV(n)| = p$.*

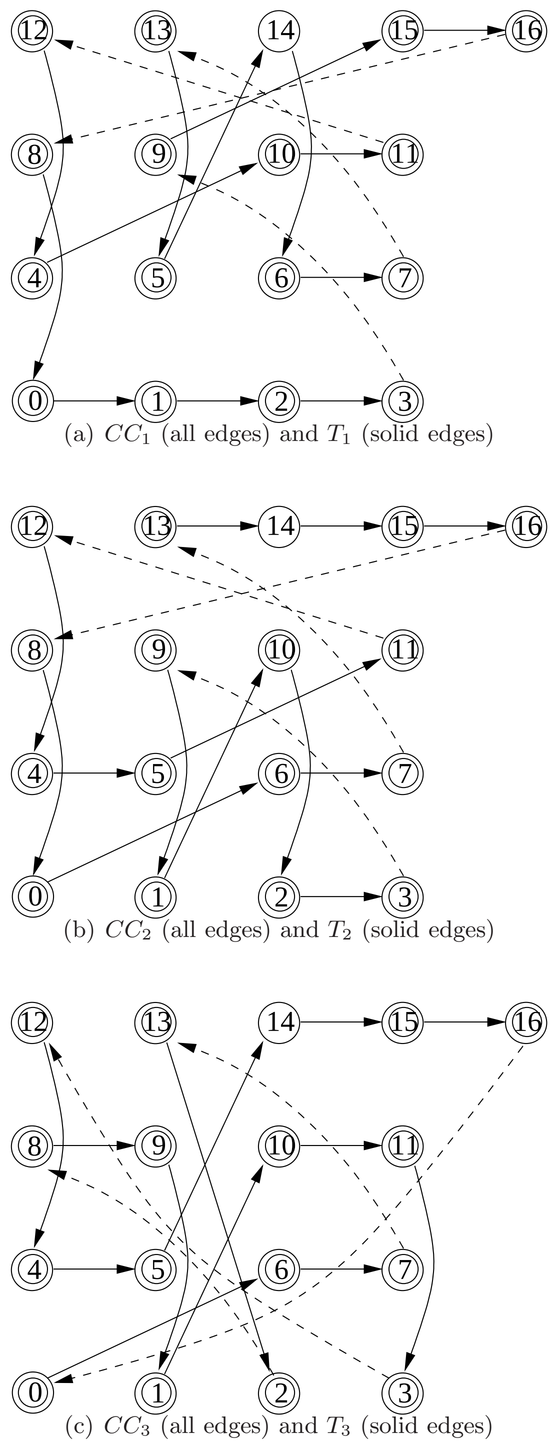
The proof is very similar to that of Lemma 4. We now continue with

Lemma 13

$$T(n) = \sum_{X \in \mathcal{P}} \beta_X T_X(n)$$

and

$$T_X(n+1) = \sum_{X' \in \mathcal{P}} \alpha_{X,X'} T_X(n).$$

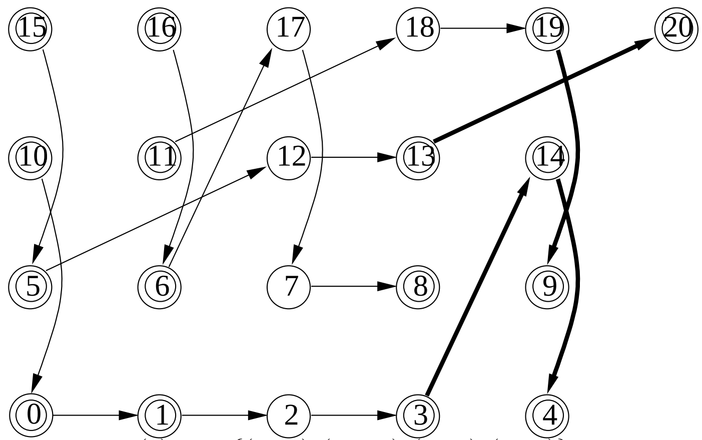


(a) CC_1 (all edges) and T_1 (solid edges)

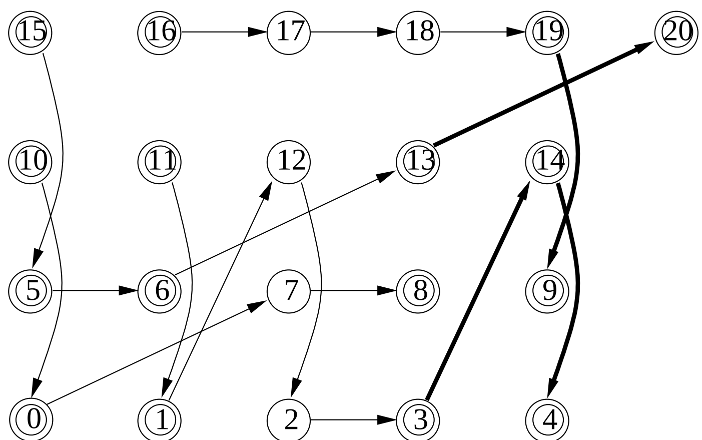
(b) CC_2 (all edges) and T_2 (solid edges)

(c) CC_3 (all edges) and T_3 (solid edges)

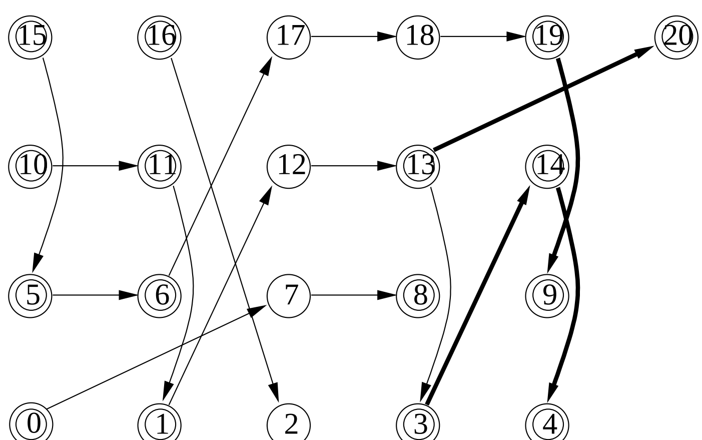
Figure 7: All of the figures are subsets of $C_{4n+1}^{1,n+2,2n+1}$. Solid edges are in L_n; dashed edges are in $\texttt{Hook}(n)$. The union of solid and dashed edges comprise different cycle covers CC_i, $i = 1, 2, 3$ in $C_{4n+1}^{1,n+2,2n+1}$. Removing the dashed edges leaves three legal covers T_i, $i = 1, 2, 3$ in L_n. $C(T_1) = C(T_2) = ((1, 1, 1, 1, 0, 0, 0, 0), (0, 1, 0, 1, 0, 1, 0, 1))$. $C(T_3) = ((0, 1, 1, 1, 0, 1, 0, 0), (0, 0, 0, 1, 1, 1, 0, 1))$.



(a) $T_1 \cup \{(3,14),(13,20),(14,4),(19,9)\}$



(b) $T_2 \cup \{(3,14),(13,20),(14,4),(19,9)\}$



(c) $T_3 \cup \{(3,14),(13,20),(14,4),(19,9)\}$

Figure 8: n was increased from 4 to 5 and $S = \{(3,14),(13,20),(14,4),(19,9)\}$ was added to T_i, of previous figure. Note that, $C(T_3 \cup S) = \emptyset$ since it is no longer a legal cover (see vertex 13). $C(T_1 \cup S) = C(T_2 \cup S) = ((1,1,1,1,0,0,0,0),(0,1,0,0,1,1,0,1))$.

We reuse the concept of consistent ordering introduced in Definition 10. It is now straightforward to redo the steps of the proof of Lemma 6 to prove

Lemma 14 *Let $A = (\alpha_{X,X'})$. If $X \in \mathcal{P}$ is ordered consistently, then there exists an $2^{p\bar{s}} \times 2^{p\bar{s}}$ matrix $\bar{A}$ such that,*

$$\begin{pmatrix} \bar{A} & 0 & \cdots & 0 \\ 0 & \bar{A} & 0 & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{A} \end{pmatrix}$$

i.e. $A = diag(\bar{A}, \bar{A}, \dots, \bar{A})$ where A contains $2^{p\bar{s}}$ copies of $\bar{A}$ on its diagonal.

To see that this really is a tight generalization of Lemma 6 note that in the constant case $p = 1$ and $s = 0$ and Lemma 14 then says that the size of $\bar{A}$ is $2^{\bar{s}}$ which is exactly the result in Lemma 6.

The main difference between the constant-jump and non-constant jump case is that, in the constant-jump case we were able, in Lemma 7, to reduce the order of the recurrence relation from the size of $\bar{A}$ to one less than the size of $\bar{A}$. This was done by using special structural properties of $\bar{A}$. One of the facts that implicitly contributed to these properties was that the *size* of $NV(n)$, i.e., the number of new vertices added at each step, was equal to one. This is not true in the non-constant jump case and we are therefore not able to extend Lemma 7 here. So, the best that we can get, from Lemma 14 is that the recurrence relation $T(n)$ satisfies a degree-$(2^{p\bar{s}})$ polynomial, an improvement of a factor of $2^{p\bar{s}}$ over the naive solution.

For an example of such a recurrence relation, see the second set of graphs in Table 1.

5 Variations and Extensions

In this section we sketch some extensions to the result in the paper as well as some other uses of the transfer matrix technique presented. For clarity, the results are only shown for constant-jump circulants. Using the techniques of Section 4 is straightforward to generalize the results in this section to non-constant jump circulants as well.

5.1 Weighted Circulants

Until now we have assumed that our circulants are 0-1 matrices corresponding to being the adjacency matrices of circulant graphs. The permanents then counted the number of cycle covers in the corresponding circulant graphs. An obvious generalization is to permit the nonzero a_i to be arbitrary values.

In this case the matrix becomes a weighted adjacency matrix. For subsets T of the edges in C_n let *weight* of T be $w(T) = \prod_{(i,j)\in T} a_{i,j}$. In this case the permanent is the sum of the weights of all cycle covers in the corresponding

C_n, i.e., $T(n) = \sum_{T \in \mathcal{CC}_n} w(T)$. We can modify our technique by changing the definition of $T_x(n)$ in Definition 8 to

$$T_X(n) = \sum_{T \in \mathcal{L}_X(n)} w(T)$$

and the definitions of $\beta_{X,S}$ and $\alpha_{X,X',S'}$ in Definition 9 to be

$$\beta_{X,S} = \begin{cases} w(S) & \text{if } X \cup S \text{ is a cycle cover} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\alpha_{X,X',S'} = \begin{cases} w(S) & \text{if } C(X' \cup S') = X \\ 0 & \text{otherwise} \end{cases} .$$

With these changes the rest of the derivations and analyses remain the same and all of the Lemmas and proofs follow accordingly. In particular, we can show that the permanent still satisfies a degree($2^{\bar{s}} - 1$) recurrence relation.

5.2 Counting Cycles in Restricted Permutations

In Section 1 we discussed how the permanent evaluates the *number* of restricted permutations using the given jumps, i.e., $T(n)$ also counted the number of permutations in

$$S_n(S) = \{\pi \in S_n \,:\, \pi[i] - i \bmod (n) \in S\}.$$

We can easily modify the transfer matrix technique to answer other questions about these permutations. As an example, suppose that we pick a permutation π uniformly at random from $S_n(S)$ and set $X = \#$ of cycles in π. What can be said about the moments of X?

First assume that, as previously, $0 = s_1 < s_2 < \cdots < s_k$. Suppose now that for cycle cover $T \in \mathcal{CC}(n)$ we define $\#_C(T)$ to be the *number* of cycles composing cover T and set

$$TC_i(n) = \sum_{T \in \mathcal{CC}(n)} \left(\#_C(T)\right)^i .$$

That is, $TC_0(n) = T(n)$ while $TC_1(n)$ is the total number of cycles summed over all cycle-covers in C_n. Then, again by the correspondence, we have that the moments of X are given by

$$\forall i \geq 0, \quad E(X^i) = \frac{TC_i(n)}{TC_0(n)}.$$

The interesting point is that the transfer matrix approach introduced in this paper can mechanically be extended through appropriate changes to the definition of $T_X(n)$ in Definition 8 and the definitions of $\beta_{X,S}$ and $\alpha_{X,X',S'}$ in Definition 9, to permit showing that for every i, $TC_i(n)$ satisfies a fixed-order constant coefficient recurrence relation. For given, $s_1, s_2, \ldots, s_k$ this permits, for example, calculating $E(X)$ and $Var(X)$.

We should note that we are only saying that for the cost functions, $TC_i(n)$, the transfer matrix defined by Lemma 5 exists. Lemmas 6 and 7 will no longer hold, though. So the degree of the recurrence relation will be $2^{2\bar{s}}$ and not $2^{\bar{s}}-1$..

Another complication is that in the general case $TC_i(n)$, $i > 0$, we may no longer assume that $0 = s_1 < s_2 < \cdots < s_k$. Recall that we were allowed to make this assumption when calculating the permanent ($i = 0$) because the permanent was invariant under rotation of rows. This is no longer true for $TC_i(n)$, $i > 0$. As an example, consider the simple circulants C_n^0 (every vertex points to itself) and C_n^1(every vertex points to its neighbor). The adjacency matrix of the first is I_n; the adjacency matrix of the second P_n^1. These T are rotationally equivalent to each other. In both cases there is only one cycle cover; in C_n^0 it is the union of n self loops; in C_n^1 the directed circle. So, for C_n^0, $TC_1(n) = 1$ while for C_n^1, $TC_1(n) = n$ and their values are different. Thus, rotationally equivalent circulant matrices may have different values of $TC_1(n)$.

We therefore need to modify our technique to work when $0 \neq s_1$ by appropriately modifying the definition of classifications. The major new complication here is that some of the edges in $\texttt{Hook}(n)$ might be going from $L(n)$ to $R(n)$ rather than from $R(n)$ to $L(n)$. Set

$$\begin{array}{rclrcl} S^+ & = & \{s \in S : s \geq 0\}, & s^+ & = & \max_{s\in S^+} s, \\ S^- & = & \{s \in S : s < 0\}, & s^- & = & \max_{s\in S^-} |s| \text{ (if } S^- = \emptyset \text{ set } s^- = 0) \end{array}$$

Now define

$$\begin{array}{rclrcl} L^+(n) & = & \{0, \dots s^+ - 1\}, & R^+(n) & = & \{n - s^+, \dots, n-1\}, \\ L^-(n) & = & \{0, \dots s^- - 1\}, & R^-(n) & = & \{n - s^-, \dots, n-1\}. \end{array}$$

Set $\bar{s} = s^+ + s^-$ and let $\mathcal{P}$ be the set of $2^{2\bar{s}}$ tuples (L_+, L_-, R_+, R_-) where L_+, L_-, R_+, R_- are, respectively, binary s^+, s^-, s^+, s^- tuples.

Definition 17 *$T \subseteq E_L(n)$ is a* legal cover *of L_n if*

- $\forall v \in V, \quad \texttt{ID}_T(v) \leq 1$ *and* $\texttt{OD}_T(v) \leq 1$.
- $\forall v \in V - (L^+(n) \cup R^-(n)), \quad \texttt{ID}_T(v) = 1.$
- $\forall v \in V - (L^-(n) \cup R^+(n)), \quad \texttt{OD}_T(v) = 1.$

Let T be a *legal-cover* of L_n. The *classification of T* will now be $C(T) = (L_+^T, L_-^T, R_+^T, R_-^T) \in \mathcal{P}$ where

$$\begin{array}{rrcl} \forall 0 \leq i < s^+, & L_+^T(i) & = & \texttt{ID}_T(i) \\ & R_+^T(i) & = & \texttt{OD}_T(n-1-i), \\ \forall 0 \leq i < s^-, & R_-^T(i) & = & \texttt{ID}_T(n-1-i), \\ & L_-^T(i) & = & \texttt{OD}_T(i). \end{array}$$

Not that the difference between this and the previously defined classifications was that previously, because $s_0 = 0$, we had $L^-(n) = R^-(n) = \emptyset$. Given these

$C_n^{-1,0,1}$	$TC_1(n) = 3TC_1(n-1) - TC_1(n-2)$ $-3TC_1(n-3) + TC_1(n-4)$ $+TC_1(n-5)$ initial values $22, 42, 80, 149, 274$ for $n = 4, 5, 6, 7, 8$	$TC_1(n)$ $\sim \frac{\phi^4}{\phi^2+\phi^4} n\phi^n$ $\sim .7236n\phi^n$	$\frac{TC_1(n)}{TC_0(n)} \sim .7236n$
$C_n^{0,1,2}$	$TC_1(n) = 3TC_1(n-1) - 6TC_1(n-3)$ $+2TC_1(n-4) + 4TC_1(n-5)$ $-TC_1(n-6) - TC_1(n-7)$ initial values $21, 32, 56, 93, 161, 275, 475$ for $n = 4, 5, \ldots, 10$	$TC_1(n)$ $\sim \frac{\phi^2}{\phi^2+\phi^4} n\phi^n$ $\sim .2764n\phi^n$	$\frac{TC_1(n)}{TC_0(n)} \sim .2764n$

Table 2: $TC_1(n)$ is the number of cycles summed over all cycle covers in the given graph with n vertices. $TC_0(n) = T(n)$ is the number of cycle covers. In Table 1 we saw that, in both cases, $TC_0(n) \sim \phi^n$ where $\phi = (1+\sqrt{5})/2$.

new definitions, we can use the same transfer matrix machinery as before to derive recurrence relations for the $TC_i(n)$.

As an illustration recall the results from Table 1 counting the number of cycle covers in $C_n^{-1,0,1}$ and $C_n^{0,1,2}$. Even though these two graphs are *not isomorphic* they had the same number of cycle-covers because the adjacency matrix of the second is just the adjacency matrix of the first with every row (cyclically) shifted over one step. Since permanents are invariant under cyclic shifts both matrices have the same permanent which is $\sim \phi^n$ where $\phi = (1+\sqrt{5})/2$.

We calculated $TC_1(n)$ for both cases with the results given in Table 2. In both cases we have that $TC_1(n) \sim cn\phi^n$. This means that if a permutation on n items is chosen at random from the corresponding distribution then, on average, it will have $\frac{T_1(n)}{T_0(n)} \sim cn$ cycles. It is interesting to note that that c is different for the two cases.

5.3 Hamiltonian Cycles and Other Problems

Finally, we note that a minor modification to the transfer-matrix technique permits using it to show that the number of *Hamiltonian Cycles* in a directed circulant graph C_n also satisfies a constant-coefficient recurrence relation in n. This fact was previously known for *undirected* circulant graphs [9, 10] but doesn't seem to have been known for directed circulants, with the exception of the special case of in(out)-degree 2 circulants [21], also known as *two-stripe* circulants.

Again, as when calculating $TC_1(n)$ in the previous subsection, we may no longer assume that $s_1 = 0$. We reuse the definitions of $L^+(n), L^-(n), R^+(n), R^-(n)$ introduced above and define a

Definition 18 *$T \subseteq E_L(n)$ is a* legal tour *of L_n if (i) T is a Hamiltonian Cycle of L_n or (ii)*

- $\forall v \in V, \quad \mathtt{ID}_T(v) \le 1$ *and* $\mathtt{OD}_T(v) \le 1$.

- $\forall v \in V - (L^+(n) \cup R^-(n))$, $\quad \mathtt{ID}_T(v) = 1$.
- $\forall v \in V - (L^-(n) \cup R^+(n))$, $\quad \mathtt{OD}_T(v) = 1$.
- T *contains no cycles*

Note that if T is legal and is not Hamiltonian, then T is composed of paths in which (i) the start of each path is in $R^+(n) \cup L^-(n)$ (ii) the end of each path is in $L^+(n) \cup R^-(n)$ and (iii) every vertex is on exactly one path (if a vertex v is isolated we consider it to be lying on a zero-length path that starts and ends at v). The *classification* of T will then be the union of the (start,end) pairs describing the starting and ending points of each path. The number of such classifications is finite. Furthermore, the classification of $T \cup S$ where T is a legal tour and $S \subseteq \mathtt{New}(n)$ or $S \subseteq \mathtt{Hook}(n)$ depends only upon the classification of T and the edges in S. We can therefore use the method described in this paper to show that the number of Hamiltonian cycles in C_n satisfies a recurrence relation.

We conclude by noting that there is nothing particularly special about Hamiltonian Cycles and that the technique will enable counting many other structures in directed circulant graphs as well. As an example, it is not too difficult to modify the method to show that the number of Eulerian Tours in such graphs also satisfies a constant-coefficient recurrence relation in n.

6 Conclusion

In this paper we showed a new derivation of Minc's result [14, 17]) that the permanent of parametrized circulant matrices satisfies a recurrence relation. Instead of being algebraic our new technique was combinatorial. We took advantage of the fact that permanents of 0/1 matrices count the number of directed cycle covers in the matrix associated with the graph to transform the problem into a counting one. We were then able to decompose circulant matrices in such a way as to allow the use of the transfer matrix method to count the number of cycle covers. Finally, we were able to show that the transfer matrix was block diagonal, with all blocks being copies of each other, reducing the order of the characteristic polynomial of the transfer matrix (and thus of the corresponding recurrence relation for the permanents).

A benefit of this new derivation is that it easily extends to the analysis of non-constant (linear) jump circulants, something that the original Minc result could not handle. It also permits counting many other properties of circulant graphs, e.g., the number of Hamiltonian cycles.

References

[1] E. Bax and J. Franklin. A permanent algorithm with $\exp[\Omega(n^{1/3}/2\ln n)]$ expected speedup for 0-1 matrices. *Algorithmica*, 32:157–162, 2002.

[2] A. Bernasconi, B. Codenotti, V. Crespi, and G. Resta. How fast can one compute the permanent of circulant matrices? *Linear Algebra and its Applications*, 292(1-3):15–37, 1999.

[3] N. L. Biggs, R. M. Damerell, and D. A. Sands. Recursive families of graphs. *J. Combin. Theory Ser. B*, 12:123–131, 1972.

[4] B. Codenotti, V. Crespi, and G. Resta. On the permanent of certain (0, 1) toeplitz matrices. *Linear Algebra and its Applications*, 267:65–100, 1997.

[5] B. Codenotti and G. Resta. On the permanent of certain circulant matrices. *Algebraic combinatorics and computer science*, pages 513–532, 2001.

[6] B. Codenotti and G. Resta. Computation of sparse circulant permanents via determinants. *Linear Algebra and its Applications*, 355(1-3):15–34, 2002.

[7] B. Codenotti, J. D. Shparlinski, and A. Winterhof. On the hardness of approximating the permanent of structured matrices. *Computational Complexity*, 11(3-4):158–170, 2002.

[8] P. Dagum, M. Luby, M. Mihail, and U. Vazirani. Polytopes, permanents and graphs with large factors. In *Proceedings of the Twentyninth IEEE Symposium on Foundations of Computer Science*, pages 412–421, 1988.

[9] M. J. Golin and Y. C. Leung. Unhooking circulant graphs: A combinatorial method for counting spanning trees and other parameters. In *Proceedings of the* 30'th International Workshop on Graph-Theoretic Concepts in Computer Science, pages 296–307, 2004.

[10] M. J. Golin, Y. C. Leung, and Y. J. Wang. Counting spanning trees and other structures in non constant-jump circulant graphs. In *The 15th Annual International Symposium on Algorithms and Computation*, pages 508–521, 2004.

[11] Alston S. Householder. *The Theory of Matrices in Numerical Analysis*. Dover, 1975.

[12] M. Jerrum, A. Sinclair, and E. Vigoda. A polynomial-time approximation algorithm for the permanent of a matrix with nonnegative entries. *Journal of the ACM*, 51(4):671–697, 2004.

[13] N. Metropolis, M. L. Stein, and P. R. Stein. Permanents of cyclic (0, 1) matrices. *J. Combin. Theory Ser. B*, 7:291–321, 1969.

[14] H. Minc. *Permanents*, volume 6 of *Encyclopedia of mathematics and its applications*. Addison-Wesley Pub. Co., 1978.

[15] H. Minc. Recurrence formulas for permanents of (0,1)-circulants. *Linear Algebra and its Applications*, 71:241–265, 1985.

[16] H. Minc. Permanental compounds and permanents of (0,1)-circulants. *Linear Algebra and its Applications*, 86:11–42, 1987.

[17] H. Minc. Theory of permanents. *Linear and Multilinear Algebra*, 21(2):109–148, 1987.

[18] M. Noy and A. Ribó. Recursively constructible families of graphs. *Advances in Applied Mathematics*, 32:350–363, 2004.

[19] R. P. Stanley. *Enumerative combinatorics.* Wadsworth & Brooks/Cole Advanced Books & Software, Monterey, Calif., 1986.

[20] L. G. Valiant. The complexity of enumeration and reliability problems. *SIAM J. Comput*, 8:410–421, 1979.

[21] Q. F. Yang, R. E. Burkard, E. Cela, and G. Woeginger. Hamiltonian cycles in circulant digraphs with two stripes. *Discrete Math*, 176:233–254, 1997.

A A Worked Example for $C_n^{0,1,2}$

In Sections 2 and 3 we derived that $T(n)$, the number of cycle covers in $C_n^{0,1,2}$, satisfies

$$\forall n \geq 2\bar{s}, \quad T(n) = \beta\,\bar{T}(n) \quad \text{and} \quad \bar{T}(n+1) = A\,\bar{T}(n)$$

where $\beta = (\beta_X)_{X \in \mathcal{P}}$ and $A = (\alpha_{X,X'})_{X,X' \in \mathcal{P}}$.

For $C_n^{0,1,2}$, $\bar{s} = 2$. Definition 8 then says that every $X \in \mathcal{P}$ is in the form $X = (L^X, R^X)$ where $L^X, R^X \in \{0,1\}^2$. We can therefore represent every X by a four-bit binary vector in which the first two bits represent L^X and the last two R^X; there are 16 such $X \in \mathcal{P}$.

Ordering the X lexicographically we calculate that β is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$\bar{T}(4)$ is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 3 & 2 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^t$$

(where the t denotes taking the transpose), and *Transfer matrix* A is

$$\left(\begin{array}{cccc|cccc|cccc|cccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)$$

The Lexicographic ordering is consistent so, as predicted by Lemma 7, A is partitioned into 16 4×4 blocks where all but the diagonal blocks are 0 and all of the diagonal blocks are equal to some 4×4 matrix $\bar{A}$ which in this case is

$$\bar{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that the lexicographic ordering on four-bit vectors also has the property that

$$(0,0) < (0,1) < (1,0) < (1,1).$$

This means that if $X_1 = (L_1, X_1)$, $X_2 = (L_2, R_2)$ and $L_1 = L_2$ then if the number of '0's in R_1 is less than the number of '0's in R_2 then $X_1 < X_2$. This satisfies the conditions of the ordering used in the proof of Lemma 7 which then implies that $\bar{A}$ should be in the form

$$\bar{A} = \begin{pmatrix} B_0 & 0 & 0 \\ 0 & B_1 & 1 \\ 0 & 1 & B_2 \end{pmatrix}$$

where B_i is a $\binom{2}{i} \times \binom{2}{i}$ matrix. We do observe this behavior with $B_0 = B_2 = (1)$ and $B_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. The characteristic polynomial of B_0 and B_2 is $P_0(x) = x - 1$. The characteristic polynomial of B_1 is $P_1(x) = x^2 - x - 1$.

This implies that

$$Q(x) = P_1(x)P_0(x) = (x^2 - x - 1)(x - 1) = x^3 - 2x^2 + 1$$

annihilates A.

Working through the details we can then solve to find that, for $C_n^{0,1,2}$, $T(n) = 2T(n-1) - T(n-3)$ with initial values $T(4) = 9$, $T(5) = 13$, and $T(6) = 12$.

B Proofs of Lemmas 8 and 9

Proof of Lemma 8:

We only prove the $\Rightarrow$ part. The reverse direction can be proved by the same argument.

(a) If $((u_1, v_1), (u_2, v_2)) \in \widehat{E}_L(n)$, there exist i such that

$$\begin{aligned} f(n; u_2, v_2) - f(n; u_1, v_1) &= p_i n + s_i \bmod (pn + s) \\ &\text{and} \\ u_2 - u_1 &= p_i \bmod p. \end{aligned}$$

If $f(n; u_2, v_2) \geq f(n; u_1, v_1)$, then

$$f(n; u_2, v_2) - f(n; u_1, v_1) = p_i n + s_i \text{ and } u_2 - u_1 = p_i.$$

When n is increased to $n + 1$,

$$\begin{aligned} & f(n+1; u_2, v_2) - f(n+1; u_1, v_1) \\ = \;& u_2(n+1) + v_2 - u_1(n+1) - v_1 \\ = \;& (u_2 n + v_2 - u_1 n - v_1) + (u_2 - u_1) \\ = \;& p_i(n+1) + s_i \end{aligned}$$

(b) If $f(n; u_2, v_2) < f(n; u_1, v_1)$, then

$$pn + s + f(n; u_2, v_2) - f(n; u_1, v_1) = p_i n + s_i \text{ and } p + u_2 - u_1 = p_i.$$

When n is increased to $n + 1$,

$$\begin{aligned} & p(n+1) + s + f(n+1; u_2, v_2) - f(n+1; u_1, v_1) \\ = \;& p(n+1) + s + u_2(n+1) + v_2 - u_1(n+1) - v_1 \\ = \;& (pn + s + u_2 n + v_2 - u_1 n - v_1) + (p + u_2 - u_1) \\ = \;& p_i(n+1) + s_i \end{aligned}$$

Therefore, in both cases, $((u_1, v_1), (u_2, v_2)) \in \widehat{E}_L(n + 1)$. □

Proof of Lemma 9:

We split the proof into two parts.
(a) $\texttt{Hook}(n) \subseteq R(n) \times L(n)$:

Let $e = ((u_1, v_1), (u_2, v_2))$ be an edge in $\widehat{E}_C(n)$ associated with the jump $p_i n + s_i$. Note that $e \in \texttt{Hook}(n)$ if and only if

$$f(n; u_2, v_2) - f(n; u_1, v_1) = p_i n + s_i \bmod pn + s \text{ and } u_2 - u_1 \neq p_i \bmod p$$

There are two cases:

(i) $f(n; u_1, v_1) \leq f(n; u_2, v_2)$.

$$\begin{aligned} &\Rightarrow \;\; u_2 n + v_2 = u_1 n + v_1 + p_i n + s_i \\ &\Rightarrow \;\; (u_2 - u_1 - p_i)n = s_i + v_1 - v_2 \end{aligned}$$

$u_2 - u_1 \neq p_i \bmod p$ implies $u_2 - u_1 - p_i \neq 0$. On the other hand,

$$-2n < s_i + v_1 - v_2 < 2n,$$

hence $u_2 - u_1 - p_i = \pm 1$.

If $u_2 - u_1 - p_i = 1$, then $s_i + v_1 - v_2 = n$, so $v_1 \geq n - s_i \geq n - \bar{s}$. Since, by definition, $v_1 \leq n - 1$, we also have $v_2 \leq s_i - 1$. Furthermore, $u_1 = u_2 - p_i - 1 < p - 1$. Hence $e \in R(n) \times L(n)$.

If $u_2 - u_1 - p_i = -1$, this implies $s_i + v_1 - v_2 + n = 0$. We then have $v_2 \geq n + s_i \geq n + s$, which is not possible since it's outside the range of v_2.

(ii) $f(n; u_1, v_1) > f(n; u_2, v_2)$.

$$\begin{aligned} &\Rightarrow \quad u_2 n + v_2 + pn + s = u_1 n + v_1 + p_i n + s_i \\ &\Rightarrow \quad (u_2 - u_1 + p - p_i)n = s_i - s - v_2 + v_1. \end{aligned}$$

Similar to the previous case:

$$-2n < s_i - s - v_2 + v_1 < 2n,$$

and $u_2 - u_1 \neq p_i \bmod p$ implies $u_2 - u_1 + p - p_i = \pm 1$.

If $u_2 - u_1 + p - p_i = 1$, then $s_i - s - v_2 + v_1 = n$. Thus $v_1 \geq n + s - s_i$, and $v_2 \leq s_i - 1$. Hence $e \in R(n) \times L(n)$.

If $u_2 - u_1 + p - p_i = -1$, this implies $s_i - s - v_2 + v_1 + n = 0$. We then have $u_2 = u_1 + p_i - p - 1 \leq p_i - 2 < p - 1$ which implies $v_2 \leq n - 1$. However this results in $v_2 = s_i - s + n + v_1 \geq n + s_i - s \geq n$ which is not possible since it's outside the range of v_2.

Therefore, $\mathtt{Hook}(n) \subseteq (R(n) \times L(n))$.

(b) $\underline{\mathtt{New}(n) \subseteq (R(n) \times NV(n)) \cup (NV(n) \times NV(n)):}$

From Lemma 8

$$\mathtt{New}(n) \subseteq (\widehat{V}(n) \times NV(n)) \cup (NV(n) \times \widehat{V}(n)) \cup (NV(n) \times NV(n)).$$

Let $e = ((u_1, v_1), (u_2, v_2)) \in \mathtt{New}(n)$ be associated with jump $p_i(n+1) + s_i$.

First assume $(u_1, v_1) \in \widehat{V}(n)$ Consider the edge e' starting with (u_1, v_1) associated with jump $p_i n + s_i$ in $\widehat{E}_C(n)$, i.e., in circulant graph C_n and not lattice graph L_{n+1}. Then

$$e \in \mathtt{New}(n) \Rightarrow e \notin \widehat{E}_L(n) \Rightarrow e' \in \mathtt{Hook}(n).$$

So, from part (a), (u_1, v_1) is in $R(n)$. Because $(u_2, v_2) \in NV(n)$, $((u_1, v_1), (u_2, v_2)) \in R(n) \times NV(n)$.

Now assume that $(u_2, v_2) \in \widehat{V}(n)$. Consider the edge e' ending with (u_2, v_2) associated with jump $p_i n + s_i$ in $\widehat{E}_C(n)$. Again

$$e \in \mathtt{New}(n) \Rightarrow e \notin \widehat{E}_L(n) \Rightarrow e \in \mathtt{Hook}(n).$$

So, from part (a), $(u_1, v_1) \in L(n)$. However $L(n) \cap NV(n) = \emptyset$, so such a (u_1, v_1) does not exist. $\square$